# When is a Risky Asset "Urgently Needed"?

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**Abstract**

The demand for commodities in standard applications typically is increasing in income, whereas the demand for the risk free asset in the classic portfolio problem often decreases with income. The latter is shown to occur if and only if the consumer's uncertainty preferences over assets satisfy the condition that the risk free asset is more readily substituted for the risky asset as the quantity of the risky asset increases. In this case, the risky asset is said to be "urgently needed" following the terminology of Johnson in his classic 1913 certainty analysis [19]. The asset and certainty settings differ in critical ways which result in a much greater likelihood for the urgently needed preference property to be satisfied in the portfolio problem. We provide several sufficient conditions for when the risky asset will be urgently needed and a surprisingly simple, complete characterization for widely popular members of the HARA (hyperbolic absolute risk aversion) class. For more general preferences, two examples are given where it is possible to fully describe the region of asset space in which the risky asset is urgently needed. Finally, using a standard representative agent model we show that the risky asset being urgently needed is equivalent to the equilibrium (relative) price of the risky asset increasing with its own supply. *JEL* Codes: D01, D11, D53.

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# 1 Introduction

While the possibility of a good being inferior is discussed in every introductory economics class, it turns out that for most utility functions used in practice the demand for each commodity actually increases with income. However in the classic single period portfolio model with one risky asset and one risk free asset, we have recently shown in [23] that the demand for the risk free asset can decrease with income (and the risk free asset can even be a Giffen good). Moreover, this can occur for perfectly standard forms of uncertainty preferences such as members of the widely popular HARA (hyperbolic absolute risk aversion) family of utility functions.[1]

To explain why this significant difference between the commodity and asset models arises, we show that although in both cases the necessary and sufficient condition for demand to decrease with income corresponds to the same indifference curve property, the likelihood of this condition being satisfied is much greater in the asset setting. Assuming classic Expected Utility preferences over end of period wealth, the demand for the risk free asset will decrease in income if and only if the MRS (marginal rate of substitution) between the risky and risk free assets increases as the quantity of the risky asset increases. Along the Expected Utility indifference curve associated with the greater quantity of the risky asset, the consumer is willing to give up relatively more of the risk free asset that has been held fixed to obtain more of the risky asset. As a result, we refer to the risky asset as being "urgently needed" – following the terminology introduced by Johnson [19] in his remarkable 1913 certainty demand analysis.

There are three critical differences between the commodity and asset settings when considering whether a commodity or an asset is urgently needed. First, Expected Utility which is concave can never be supermodular in assets although preferences over commodities are typically assumed to be concave and supermodular.[2] As a result, the cross partial derivative of the Expected Utility function with respect to the risky and risk free asset holdings is shown to always be negative, satisfying a necessary condition for the MRS to increase with the quantity of the risky asset. Second whereas typically assumed commodity preferences for two goods are essentially symmetric, the induced utility for assets is far from symmetric. Indeed the standard assumption of decreasing absolute risk aversion ensures that the demand for the risky asset always increases with income, but does not imply that the risk free asset behaves in the same way. The third difference is that commodity demands are required to be positive whereas it is standard to allow negative holdings (or short-selling) of the risk free asset.

In characterizing when the risky asset is urgently needed, we provide two general sufficient

[1] See Gollier [12] for a description of the HARA family of utility functions.

[2] See Chambers and Echenique [7].

conditions in which the quantity of the risk free asset plays a surprisingly important role. The first condition involves the (Arrow-Pratt) measures of absolute and relative risk aversion and the sign of the risk free asset holdings. The second requires preferences to satisfy the Inada conditions ensuring that the risk free asset Engel curve begins at its origin. For the HARA class of Expected Utility preferences as well as for all homothetic preferences, we derive necessary and sufficient conditions, which very surprisingly depend only on the quantity of the risk free asset. To bridge the gap between the latter conditions and the more general sufficient conditions we analyze several examples in which it is possible to fully characterize regions of asset space where the risky asset is and is not urgently needed.

The risky asset being urgently needed also has strong implications for equilibrium prices. In a standard representative agent exchange economy, we show that the risky asset's (relative) equilibrium price increases with its supply if and only if the asset is urgently needed.[3] For the special case of HARA preferences, this result is seen to depend solely on the quantity of the assumed supply of the risk free asset.

It should be noted that in our earlier paper [23], we demonstrated that for quite standard forms of Expected Utility the risk free asset can be an inferior good and more remarkably a Giffen good. Given that the slope of the risk free asset Engel curve played an important role in determining inferior good behavior,[4] we derived a necessary and sufficient condition to determine the sign of its slope. But this condition is not particularly intuitive and is computationally difficult to verify in practice because it is based on the restrictive complete market assumption. In this paper to explain why demand can more readily decrease with income for assets than commodities, we utilize the more intuitive urgently needed indifference curve property which is not based on complete markets and can be used to directly compare the asset and commodity cases.

The rest of the paper is organized as follows. In Section 2, we review the certainty analysis of Johnson and provide a concrete example illustrating the relationship between one good becoming urgently needed and the demand of a second good decreasing with income. Section 3 investigates when the risky asset becomes urgently needed and compares the conditions with those in the certainty case. In Section 4, we discuss the equilibrium implication of our results. The final Section contains concluding comments.

[3] In certainty equilibrium analyses, the supply of each good is assumed to be positive. However, in uncertainty setting although the risky asset is typically assumed to be in positive supply, the risk free asset is most often assumed to have zero net supply (e.g., [3]). Recently a number of papers have relaxed this assumption, allowing aggregate supply to be positive (e.g., [15], [9] and [29]) or negative (e.g., [11]). While the motivation for these different supply assumptions is unrelated to our analysis, the implications of the assumptions for equilibrium price comparative statics are far from innocuous.

[4] As we argue below, the fact that the risk free asset need not be held long implies that its demand decreasing with income is not equivalent to it being an inferior good.

# 2 Urgently Needed Good: Certainty Case

In certainty settings, the possibility that demand decreases with increasing income is typically precluded by standard preference assumptions such as additive separability (or the weaker property of supermodularity (see [7])) and concavity. Indeed finding otherwise well behaved utility functions that generate such behavior is often viewed as being difficult. In this Section, we first review the preference characterization of inferior good behavior derived in Johnson's classic paper [19] and then provide an example which satisfies his condition. In the next Section we show how application of the Johnson result in an uncertainty setting differs in several critical ways from the certainty case.

Consider a single period, two good setting in which $x$ and $y$ denote the units of the goods. Assume a consumer whose preferences over $(x, y)$ pairs defined on a convex subset $\Omega$ of the positive orthant are representable by a strictly quasiconcave utility $U(x, y)$ which is increasing in each good, and satisfies $U \in C^3$. The consumer can be viewed as solving the optimization problem

$$\max_{x,y} U(x, y) \tag{1}$$

subject to

$$I = p_x x + p_y y, \tag{2}$$

where $p_x$ and $p_y$ denote the prices of the goods and $I$ is initial income or wealth. As is standard, $y$ is said to be an inferior good if and only if $\partial y/\partial I < 0$. Define the MRS by $\frac{U_x}{U_y}$.[5] Then we have the following result.

**Proposition 1** *(Johnson [19]) Assume the single period optimization problem given by eqns. (1)-(2). Then*

$$\frac{\partial y}{\partial I} \lesseqqgtr 0 \Leftrightarrow \frac{\partial\left(\frac{U_x}{U_y}\right)}{\partial x} \gtreqqless 0. \tag{3}$$

The intuition for Proposition 1 can be expressed very simply in terms of the two indifference curves plotted in the $x - y$ choice space in Figure 1.[6] Minus the slope of the dashed tangent to each indifference curve corresponds to the $MRS = \frac{U_x}{U_y}$. When moving from tangency point $P$ to $Q$ in Figure 1, $x$ is increased while $y$ is held constant. Corresponding to this move, the slope of the new indifference curve is seen to become steeper and the MRS increases,

$$\frac{\partial\left(\frac{U_x}{U_y}\right)}{\partial x} = \frac{U_x}{U_y}\left(\frac{U_{xx}}{U_x} - \frac{U_{yx}}{U_y}\right) > 0, \tag{4}$$

[5] Throughout this paper partial derivatives will be denoted by subscripts. For example, we define $U_x = \frac{\partial U}{\partial x}$.

[6] See [26] for a similar discussion.

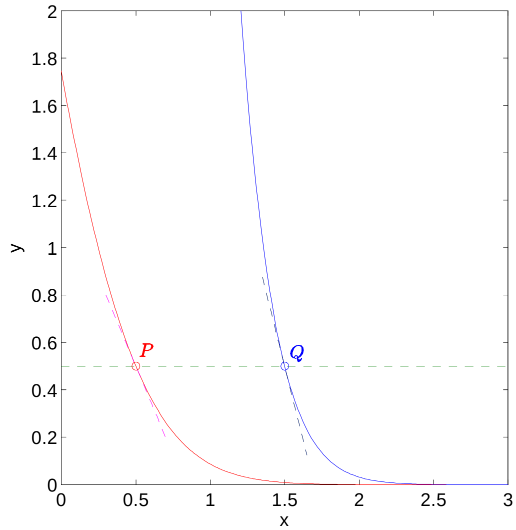



Figure 1:

and following Johnson's terminology good $x$ would be referred to as being more "urgently needed" than $y$. It follows from Proposition 1 that $\frac{\partial y}{\partial I} < 0$ and good $y$ is an inferior good.[7] Johnson viewed the opposite case from Figure 1, where the MRS decreases between the two points and $y$ is a normal good, to be the "standard" case. As good $x$ becomes relatively more abundant relative to the fixed good $y$, the consumer should be more willing to give up less of good $y$ to obtain one more unit of good $x$ along the shifted indifference curve as reflected by the decreased MRS. This is consistent with the standard assumption of supermodularity and concavity since if $U_{yx} > 0$ and $U_{xx} < 0$, then from eqn. (4) $\frac{\partial\left(\frac{U_x}{U_y}\right)}{\partial x} < 0$.

Next we consider a simple non-traditional, although well behaved form of utility, which results in good $y$ being an inferior good and $x$ being urgently needed.

**Example 1** *Consider the following strictly quasiconcave, non-supermodular utility function*

$$U\left(x,y\right) = -\frac{\left(\beta_1 x + y - a\right)^{-\delta}}{\delta} - \frac{\left(\beta_2 x + y - a\right)^{-\delta}}{\delta}, \tag{5}$$

*where $a > 0$, $\beta_1 > \beta_2 > 0$ and $\delta > -1$. It can be verified that $U$ is strictly increasing in $x$*

[7] If good $y$ is an inferior good, it follows from the budget constraint that good $x$ must be a normal good. Using the terminology of Hirshleifer [16], good $x$ can be referred to as being an ultrasuperior good since corresponding to an increase in income, the incremental demand for good $x$ not only increases corresponding to its being a normal good but increases by more than the full increase in income, which results in good $y$ becoming an inferior good, i.e., $\frac{\partial y}{\partial I} < 0$.

*and $y$ and $U_{yx} < 0$.*[8] *We have*

$$\frac{U_x}{U_y} = \frac{\beta_1 (\beta_1 x + y - a)^{-1-\delta} + \beta_2 (\beta_2 x + y - a)^{-1-\delta}}{(\beta_1 x + y - a)^{-1-\delta} + (\beta_2 x + y - a)^{-1-\delta}}. \tag{6}$$

*Since*

$$\frac{\partial \left(\frac{U_x}{U_y}\right)}{\partial x} = \frac{(1+\delta)(\beta_1 - \beta_2)(\beta_1 x + y - a)^{-2-\delta} (\beta_2 x + y - a)^{-1-\delta} \left(\frac{1}{x + \frac{y-a}{\beta_1}} - \frac{1}{x + \frac{y-a}{\beta_2}}\right)}{\left((\beta_1 x + y - a)^{-1-\delta} + (\beta_2 x + y - a)^{-1-\delta}\right)^2}, \tag{7}$$

*it follows that*

$$\frac{\partial \left(\frac{U_x}{U_y}\right)}{\partial x} \gtreqless 0 \Leftrightarrow y \lesseqgtr a, \tag{8}$$

*implying that $x$ is urgently needed (and $y$ is an inferior good) in a region of the commodity space defined by $0 < y < a$.*

Whereas the form of utility in Example 1 is non-traditional in certainty demand analysis, as we will see this form is quite standard in the uncertainty asset demand setting.

# 3 When the Risky Asset is Urgently Needed

In this Section we derive a number of restrictions on Expected Utility preferences corresponding to the risky asset being urgently needed.

## 3.1 Preliminaries

Consider a risky asset with random payoff $\widetilde{\xi} > 0$ and a corresponding arbitrary cumulative distribution function $F\left(\widetilde{\xi}\right)$, which is independent of the amount invested**.** Suppose there also exists a risk free asset with payoff $\xi_f > 0$. Let $n$ and $n_f$ denote the units of the risky asset and risk free asset, respectively. We consider portfolios consisting of a risk free asset and a risky asset where positive holdings of the former are not required. In a single period setting, the consumer's preferences are defined over random end of period wealth $\widetilde{z} = \widetilde{\xi} n + \xi_f n_f$ and satisfy the standard Expected Utility axioms where the NM (von Neumann-Morgenstern) index $W(z)$ satisfies $W \in C^3$, $W' > 0$ and $W'' < 0$. The Expected Utility function is given by

$$EW(\widetilde{z}) = EW\left(\widetilde{\xi} n + \xi_f n_f\right) = \int W\left(\widetilde{\xi} n + \xi_f n_f\right) dF\left(\widetilde{\xi}\right). \tag{9}$$

[8] It should be noted that for the utility (5) to be well-defined for all possible $\delta$, we require that $\beta_2 x + y - a > 0$. This implies that indifference curves are only defined for points in the positive region of the $x - y$ space northeast of the line $y = a - \beta_2 x$.

The consumer can be viewed as solving the optimization problem

$$\max_{n,n_f} \mathcal{W}(n, n_f) = \max_{n,n_f} EW\left(\widetilde{\xi}n + \xi_f n_f\right) \tag{10}$$

subject to

$$pn + p_f n_f = p\overline{n} + p_f \overline{n}_f, \tag{11}$$

where $\overline{n}$ and $\overline{n}_f$ denote, respectively, endowments of the risky and risk free assets and $p$ and $p_f$ are the prices of the risky and risk free assets. It will also prove convenient to define the initial income or wealth $I$ as

$$I = p\overline{n} + p_f \overline{n}_f. \tag{12}$$

The no arbitrage condition $\frac{\sup \widetilde{\xi}}{p} > \frac{\xi_f}{p_f} > \frac{\inf \widetilde{\xi}}{p}$ is assumed. Under our assumptions, it can be easily verified that the function $\mathcal{W}$ is strictly increasing and concave in both $n$ and $n_f$ and the optimization problem has a unique solution in $n$ and $n_f$. In order to guarantee non-negative wealth, a no bankruptcy condition $p\overline{n} + p_f \overline{n}_f > I_{\min}$ is assumed.[9] It will prove convenient to also assume that $\frac{E\widetilde{\xi}}{p} > \frac{\xi_f}{p_f}$, which implies that the optimal risky asset holding is positive.[10] The demand for assets is a continuous function in asset prices $(p, p_f)$ and endowments $(\overline{n}, \overline{n}_f)$. Instead of writing $n(p, p_f, \overline{n}, \overline{n}_f)$ and $n_f(p, p_f, \overline{n}, \overline{n}_f)$, we will suppress the dependence on prices and endowments whenever possible and simply use $(n, n_f)$ to denote these functions.

## 3.2 General Case

In the classic multicommodity certainty setting when demand decreases with income, a good is said to be an inferior good. However in the uncertainty portfolio case, because $n_f$ can be negative, it is necessary to modify this standard definition. Given that the conventional income effect in the Slutsky equation corresponding to $\frac{\partial n_f}{\partial p_f}$ is defined by $-n_f \frac{\partial n_f}{\partial I}$, it is natural

[9] Minimum income is defined by $I_{\min} = pn^0 + pn_f^0$, where $n^0$ and $n_f^0$ are the optimal asset holdings satisfying $\inf \widetilde{\xi} n^0 + \xi_f n_f^0 = 0$. For the complete market case, an analytical form of $I_{\min}$ is given in [23].

[10] To see that $n > 0$, note that the first order condition for the optimization problem (10)-(11) is

$$E\left[\left(\widetilde{\xi} - \frac{p}{p_f}\xi_f\right) W'\left(\widetilde{\xi}n + \xi_f n_f\right)\right] = 0.$$

Clearly, we have

$$E\left[\left(\widetilde{\xi} - \frac{p}{p_f}\xi_f\right) W'\left(\widetilde{\xi}n + \xi_f n_f\right)\right] \lesseqqgtr \left(E\widetilde{\xi} - \frac{p}{p_f}\xi_f\right) EW'\left(\widetilde{\xi}n + \xi_f n_f\right) \Leftrightarrow n \gtreqqless 0.$$

Therefore, the assumption that

$$\frac{E\widetilde{\xi}}{p} > \frac{\xi_f}{p_f}$$

implies $n > 0$.

to generalize the inferior good definition to be $n_f \frac{\partial n_f}{\partial I} < 0$.[11] When $n_f > 0$, one obtains the traditional definition $\frac{\partial n_f}{\partial I} < 0$. Alternatively when $n_f < 0$ and $\frac{\partial n_f}{\partial I} < 0$, borrowing can be viewed as being a normal good as it increases with income. We next show that, unlike the certainty case, in determining whether the MRS increases with the holdings of the risky asset, one needs to focus on whether the risk free asset Engel curve is downward sloping and not whether it is an inferior good.

In order to characterize when $\frac{\partial n_f}{\partial I} < 0$, we next extend Proposition 1 to the uncertainty portfolio setting.

**Proposition 2** *Assume the optimization problem given by eqns. (10)-(11). Then*

$$\frac{\partial n_f}{\partial I} \lesseqqgtr 0 \Leftrightarrow \frac{\partial \left( \frac{\mathcal{W}_n}{\mathcal{W}_{n_f}} \right)}{\partial n} \gtreqqless 0. \tag{13}$$

**Proof.** The first order condition for the optimization problem (10)-(11) is given by

$$\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}} = \frac{E\left[\tilde{\xi} W'\left(\tilde{\xi} n + \xi_f n_f\right)\right]}{E\left[\xi_f W'\left(\tilde{\xi} n + \xi_f n_f\right)\right]} = \frac{p}{p_f}. \tag{14}$$

Differentiating both sides of the above equation with respect to $n$ and $I$ yields, respectively,

$$\frac{\partial}{\partial n}\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right) = \frac{\mathcal{W}_{n,n}\mathcal{W}_{n_f} - \mathcal{W}_{n,n_f}\mathcal{W}_n}{\mathcal{W}_{n_f}^2} \tag{15}$$

and

$$\mathcal{W}_{n,n}\frac{\partial n}{\partial I} + \mathcal{W}_{n,n_f}\frac{\partial n_f}{\partial I} - \frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\left(\mathcal{W}_{n,n_f}\frac{\partial n}{\partial I} + \mathcal{W}_{n_f,n_f}\frac{\partial n_f}{\partial I}\right) = 0. \tag{16}$$

Combining eqns. (11) and (12), and differentiating with respect to $I$, one obtains

$$p\frac{\partial n}{\partial I} + p_f\frac{\partial n_f}{\partial I} = 1. \tag{17}$$

It follows that

$$\frac{\partial n_f}{\partial I} = -\frac{1}{p\mathcal{W}_{n_f}} \frac{\mathcal{W}_{n,n}\mathcal{W}_{n_f} - \mathcal{W}_{n,n_f}\mathcal{W}_n}{2\mathcal{W}_{n,n_f} - \frac{p}{p_f}\mathcal{W}_{n_f,n_f} - \frac{p_f}{p}\mathcal{W}_{n,n}}. \tag{18}$$

Since $\mathcal{W}(n, n_f)$ is strictly quasiconcave,

$$2\mathcal{W}_n\mathcal{W}_{n_f}\mathcal{W}_{n,n_f} - \mathcal{W}_n^2\mathcal{W}_{n_f,n_f} - \mathcal{W}_{n_f}^2\mathcal{W}_{n,n} > 0, \tag{19}$$

or equivalently

$$2\mathcal{W}_{n,n_f} - \frac{p}{p_f}\mathcal{W}_{n_f,n_f} - \frac{p_f}{p}\mathcal{W}_{n,n} > 0, \tag{20}$$

[11] This definition is consistent with that of Hicks [14] where, in a multicommodity setting, an inferior good is characterized by having a negative income elasticity. Also see [23].

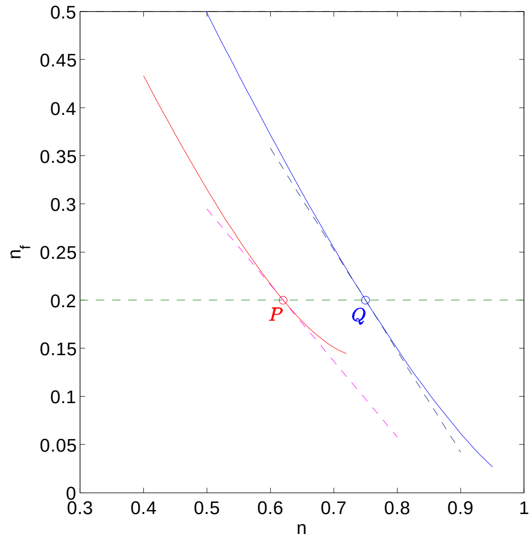



Figure 2:

implying that

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \gtreqqless 0 \Leftrightarrow \frac{\partial n_f}{\partial I} \lesseqqgtr 0. \tag{21}$$

■

**Remark 1** *Similarly, one can show that*

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n_f} \gtreqqless 0 \Leftrightarrow \frac{\partial n}{\partial I} \gtreqqless 0. \tag{22}$$

*Given the assumptions of $n > 0$ and decreasing absolute risk aversion,*[12] *it follows from Arrow [2] that $\frac{\partial n}{\partial I} > 0$. Thus from (22), we always have $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n_f} > 0$. Comparing this result with Proposition 2, there is a natural asymmetry in the behavior of the two assets in the portfolio setting. This is different from the certainty case, where there is no a priori reason to suppose the two goods are asymmetric.*

Consider Figure 2, where two Expected Utility indifference curves are plotted in the $n - n_f$ choice space and it is assumed that $n, n_f > 0$. When moving from the tangency point $P$ to $Q$ the risky asset becomes relatively more abundant and yet along the indifference curve through point $Q$, the consumer is willing to give up more of the risk free asset to obtain one more unit of the risky asset. Thus the risky asset is "more urgently needed" than the

[12]See the definition in eqn. (25) below.

risk free asset.[13] Unlike the analogous conditions in the certainty case, we will argue that $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} > 0$ and $\frac{\partial n_f}{\partial I} < 0$ should not be dismissed as being "non-standard".

Although when $n_f > 0$, the downward sloping Engel curve indicates inferior good behavior, the intuition for the risk free asset to be an inferior good is very different from that of certainty commodities. For the latter, as suggested by the expression "inferior" good, there is a long tradition of interpreting such goods as possessing inferior attributes or quality. As a result when income increases, the consumer switches to goods with superior attributes. Classic examples include substituting from potatoes to meat, from functional to stylish clothing and from basic, low cost automobiles to models with greater functionality. Gould [13] challenged the assumption that one good must be of inferior quality. He illustrates this phenomena with the case of excellent quality wine and cigars. At low levels of income, these goods may be consumed infrequently and at different times. But as income increases and the consumer seeks to enjoy both together, he may discover that increased smoking dulls the palate and interferes with the enjoyment of wine. Eventually with increasing income, the marginal utility for wine decreases with the consumption of cigars and, as a result, the demand for cigars decreases and the demand for wine increases.

In the uncertainty portfolio case it follows from

$$\frac{\partial}{\partial n}\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right) = \frac{\mathcal{W}_{n,n}\mathcal{W}_{n_f} - \mathcal{W}_{n,n_f}\mathcal{W}_n}{\mathcal{W}_{n_f}^2} \tag{23}$$

that since $\mathcal{W}_n$, $\mathcal{W}_{n_f} > 0$ and $\mathcal{W}_{n,n} < 0$, $\mathcal{W}_{n,n_f} < 0$ is a necessary condition for $\frac{\partial}{\partial n}\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right) > 0$ and $\frac{\partial n_f}{\partial I} < 0$. But given our assumption that $\widetilde{\xi}, \xi_f > 0$ and the concavity of $W$ or risk aversion of the consumer, we always have

$$\mathcal{W}_{n,n_f} = E\left[\widetilde{\xi}\xi_f W''\left(\widetilde{\xi}n + \xi_f n_f\right)\right] < 0. \tag{24}$$

Since $\mathcal{W}_{n,n_f} < 0$ is guaranteed by the assumption of risk aversion, in contrast to the certainty case it is not obvious that one good should be interpreted as possessing inferior quality or that the goods conflict as in the wine-cigar example.[14] As a result, the MRS condition

[13] In this example since $n_f > 0$, the risk free asset is an inferior good. However if in Figure 2 $n_f < 0$ and one had increasing MRS with $n$ implying $\frac{\partial n_f}{\partial I} < 0$, the risky asset would be urgently needed even though borrowing would be a normal good.

[14] It is natural to wonder what the intuition is for the cross partial derivative $\mathcal{W}_{n,n_f}$ to always be negative. First, the Expected Utility form $\mathcal{W}$ can be viewed as a concave transform of the linear asset payoff $n\widetilde{\xi} + n_f\xi_f$. This linear payoff structure is fully consistent with the intuition that the two assets are substitutes. Second, $\mathcal{W}(n, n_f)$ exhibits diminishing marginal utility in each of its arguments. Thus since $n$ and $n_f$ can be thought of as substitutes, it is natural that if one increases the quantity of one asset then the marginal utility of the other asset should decrease because it is almost like increasing the quantity of that asset. This argument is closely related to the classic notion of Edgeworth-Pareto complementarity where for the certainty utility $U(x, y)$, $U_{xy} < 0$ indicates that the goods are substitutes (see Samuelson [32]).

$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} > 0$ would seem to occur more readily for the portfolio problem versus the certainty case. But since $\mathcal{W}_{n,n_f} < 0$ is not sufficient, what else must be assumed to ensure that the risky asset is urgently needed?

As noted in Remark 1, the assumption of decreasing absolute risk aversion makes the risk free asset and risky asset asymmetric. To investigate this issue more carefully, it will prove convenient to formally introduce the classic Arrow-Pratt absolute and relative risk aversion measures

$$\tau_A(z) =_{def} -\frac{W''(z)}{W'(z)} \qquad \text{and} \qquad \tau_R(z) =_{def} -z\frac{W''(z)}{W'(z)}. \tag{25}$$

We denote the derivatives of these functions by $\tau_A'$ and $\tau_R'$, and unless stated otherwise, suppress the dependence on $z$.

Returning to Figure 2, it is obvious that when moving from point $P$ to $Q$, $\widetilde{z} = \widetilde{\xi}n + \xi_f n_f$ increases (in each state). If as a result, the MRS increases and the risky asset becomes more urgently needed implying that $\frac{\partial n}{\partial I} > 0$, then it follows from [2] that it cannot be the case that preferences satisfy $\tau_A' \geq 0$. However, decreasing absolute risk aversion is not enough for the risky asset to become more urgently needed. We next provide general sufficient conditions for when this is and is not the case and then subsequently necessary and sufficient conditions for special forms of utility.[15]

**Proposition 3** *Assume the optimization problem given by eqns. (10)-(11). If* $\tau_A' < 0$,

**(i)** $\tau_R' \leq 0$ *and* $n_f \leq 0$*, then*

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \geq 0, \tag{26}$$

**(ii)** $\tau_R' \geq 0$ *and* $n_f \geq 0$*, then*

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \leq 0, \tag{27}$$

*where for eqns. (26) and (27) the equal sign can be reached if and only if* $n_f = 0$ *and* $\tau_R' = 0$.

**Proof.** Since $W$, $W'$ and $W''$ are always defined on $\widetilde{z} = \widetilde{\xi}n + \xi_f n_f$, we will suppress the argument for simplicity. Differentiating the first order condition (14) with respect to $n$ and noticing that

$$E\left[\left(\widetilde{\xi} - \frac{p}{p_f}\xi_f\right)W'\right] = 0, \tag{28}$$

[15] Combined with Proposition 2, Proposition 3 can be viewed as a general proof of Theorem 2 in [23] where the assumption of complete markets in the latter can be dropped.

yields

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}=\frac{EW'E\left[\widetilde{\xi}\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)W''\right]}{\xi_f\left(EW'\right)^2}. \tag{29}$$

Given that $\widetilde{z}=\widetilde{\xi}n+\xi_f n_f$, we have

$$nE\left[\widetilde{\xi}\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)W''\right]=E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)W'\frac{\widetilde{z}W''}{W'}\right]-\xi_f n_f E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)W''\right]. \tag{30}$$

It follows from [12], Proposition 15 that if $\tau_A' < 0$, or equivalently, $\left(\frac{W''}{W'}\right)' > 0$, then

$$E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)W'\left(\frac{W''}{W'}\right)\right] > E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)W'\right]E\left[\frac{W''}{W'}\right]=0. \tag{31}$$

It also follows that if $\tau_R' \leq 0$, or equivalently, $\left(\frac{zW''}{W'}\right)' \geq 0$, then

$$E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)W'\frac{\widetilde{z}W''}{W'}\right] \geq E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)W'\right]E\left[\frac{\widetilde{z}W''}{W'}\right]=0. \tag{32}$$

Therefore, if $\tau_R' \leq 0$ and $n_f \leq 0$, then

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \geq 0. \tag{33}$$

Similarly one can show that if $\tau_R' \geq 0$ and $n_f \geq 0$, then

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \leq 0. \tag{34}$$

In both cases the equal sign can be reached if and only if $n_f = 0$ and $\tau_R' = 0$. ■

In Proposition 3 since the equal sign can be reached only if $n_f = 0$ and $\tau_R' = 0$, it follows from (i) that if $\tau_R' < 0$ and $n_f = 0$ we have

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} > 0. \tag{35}$$

Therefore, by continuity, there must exist a region with $n_f > 0$ where the risky asset is urgently needed for this case.

One may argue that the assumption of $\tau_R' \leq 0$ in Proposition 3 is strong.[16] Next we show that there always exists some region in $n - n_f$ space such that the risky asset becomes urgently needed if the NM index $W(z)$ satisfies the well-known Inada conditions (see [18], p.120).

[16] It should be noted that the property of decreasing relative risk aversion has received attention in empirical and experimental papers (e.g., Levy [24], Ogaki and Zhang [28], Meyer and Meyer [25], Calvet et al. [4] and [5]). Moreover, the multiperiod NM index used in standard additive habit formation models in asset pricing literatures also exhibits decreasing relative risk aversion.

**Definition 1** *A utility function $U(\mathbf{x})$ satisfies the Inada conditions if and only if $U \in C^2$, $\frac{\partial U(\mathbf{x})}{\partial x_i} > 0$, $\frac{\partial^2 U(\mathbf{x})}{\partial x_i^2} < 0$, $\lim_{x_i \to 0} \frac{\partial U(\mathbf{x})}{\partial x_i} = \infty$ and $\lim_{x_i \to \infty} \frac{\partial U(\mathbf{x})}{\partial x_i} = 0$.*

It can be easily seen that the Expected Utility $EW(\widetilde{z})$ satisfies the Inada conditions if and only if $W \in C^2$, $W' > 0$, $W'' < 0$, $W'(0) = \infty$ and $W'(\infty) = 0$.

**Proposition 4** *Assume the optimization problem given by eqns. (10)-(11). If $W(z)$ satisfies the Inada conditions, then there always exists some region in $n - n_f$ space such that $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} > 0$.*

**Proof.** If we can show that for $W(z)$ satisfying the Inada conditions, there always exists some $(p_f, I)$ such that $\partial n_f / \partial I < 0$, then Proposition 4 follows immediately from Proposition 2. The first order condition is given by

$$\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}} = \frac{E\left[\widetilde{\xi} W'\left(\widetilde{\xi} n + \xi_f n_f\right)\right]}{E\left[\xi_f W'\left(\widetilde{\xi} n + \xi_f n_f\right)\right]} = \frac{p}{p_f}. \tag{36}$$

When $n = 0$, we have

$$\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}} = \frac{E\left[\widetilde{\xi} W'\left(\xi_f n_f\right)\right]}{E\left[\xi_f W'\left(\xi_f n_f\right)\right]} \tag{37}$$

and if $W'\left(\xi_f n_f\right)$ is a positive finite number, then

$$\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}} = \frac{E\widetilde{\xi}}{\xi_f} > \frac{p}{p_f}, \tag{38}$$

implying that the first order condition cannot be satisfied. Therefore, we must have $W'\left(\xi_f n_f\right) = 0$ or $\infty$ when $n = 0$. Since $W'(\infty) = 0$, we can conclude $\xi_f n_f = 0$, or equivalently $n_f = 0$. The fact that $n = n_f = 0$ implies that $I = 0$ and the Engel curves for the risky and risk free assets both start from their respective origin. Assume $p_f$ is large enough such that

$$\frac{\xi_f}{p_f} \to \frac{\inf \widetilde{\xi}}{p}. \tag{39}$$

We want to argue

$$\inf \widetilde{\xi} n + \xi_f n_f \to 0. \tag{40}$$

The reason is as follows. If $W'\left(\inf \widetilde{\xi} n + \xi_f n_f\right)$ is finite, then

$$\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}} = \frac{E\left[\widetilde{\xi} W'\left(\widetilde{\xi} n + \xi_f n_f\right)\right]}{E\left[\xi_f W'\left(\widetilde{\xi} n + \xi_f n_f\right)\right]} > \frac{\inf \widetilde{\xi}}{\xi_f} \to \frac{p}{p_f}, \tag{41}$$

implying that the first order condition cannot be satisfied. Therefore, we have $W'\left(\inf \widetilde{\xi} n + \xi_f n_f\right) \to \infty$, or equivalently

$$\inf \widetilde{\xi} n + \xi_f n_f \to 0. \tag{42}$$

Since $\inf \widetilde{\xi} > 0$, $\xi_f > 0$ and $n > 0$, we must have $n_f < 0$. We have shown above that $n_f = 0$ when $I = 0$. We have also argued that if $p_f$ is large enough such that eqn. (39) holds, then $n_f < 0$. Due to continuity, we must have $\partial n_f / \partial I < 0$ for some income levels. Therefore, it follows from Proposition 2 that there always exists some region in $n - n_f$ space such that $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} > 0$. $\blacksquare$

**Remark 2** *The intuition for Proposition 4 is very clear. The Inada conditions are both necessary and sufficient for the risk free asset Engel curve to start from the origin,* $(I, n_f) = (0, 0)$. *Then if the risk free asset price is large enough, the consumer will short the risk free asset, i.e.,* $n_f < 0$. *Due to continuity, one must have* $\partial n_f / \partial I < 0$ *at low income levels, implying that there exists some region in* $n - n_f$ *space such that* $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} > 0$.

Whereas the conditions in Propositions 3 and 4 are only sufficient, we next provide necessary and sufficient conditions for four popular members of the widely assumed HARA class of utilities.

## 3.3 HARA Class

In general, one would expect the sign of $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ to depend on both $n$ and $n_f$.[17] However, it follows from Proposition 3 that, given decreasing absolute risk aversion and monotone relative risk aversion, the sign of $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ may depend just on the value of $n_f$. Next we show that this is indeed the case for four widely assumed members of the HARA class and moreover for these utilities much stronger conditions can be derived.

**Proposition 5** *Assume the optimization problem given by eqns. (10)-(11) and the NM index* $W(z)$ *is a member of the HARA class. Then*

**(i)** *if*

$$W(z) = -\frac{z^{-\delta}}{\delta}, \qquad \delta > -1, \tag{43}$$

[17] If the risk free asset always has an upward (flat, downward) sloping Engel curve, i.e., $\partial n_f / \partial I > (=, <)\ 0$, no matter what prices and income are, then it follows from Proposition 2 that we always have $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} < (=, >)\ 0$. Otherwise, since $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ is a function of $(n, n_f)$, its sign will depend on the values of $n$ and $n_f$.

*then*

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \gtreqless 0 \Leftrightarrow n_f \lesseqgtr 0, \tag{44}$$

**(ii)** *if*

$$W(z) = -\frac{(z-a)^{-\delta}}{\delta}, \qquad \delta > -1,\ a > 0, \tag{45}$$

*then*

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \gtreqless 0 \Leftrightarrow n_f \lesseqgtr \frac{a}{\xi_f}, \tag{46}$$

**(iii)** *if*

$$W(z) = -\frac{(z+a)^{-\delta}}{\delta}, \qquad \delta > -1,\ a > 0, \tag{47}$$

*then*

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \gtreqless 0 \Leftrightarrow n_f \lesseqgtr -\frac{a}{\xi_f}, \tag{48}$$

**(iv)** *if*

$$W(z) = -\frac{\exp(-\lambda z)}{\lambda}, \qquad \lambda > 0, \tag{49}$$

*then*

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} < 0. \tag{50}$$

**Proof.** We apply a similar method as in the proof of Proposition 3 which does not rely on the demand properties implied by the specific forms of HARA utility. For case (i),

$$\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}} = \frac{E\left[\widetilde{\xi}\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta}\right]}{E\left[\xi_f\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta}\right]}. \tag{51}$$

Therefore,

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = \frac{(1+\delta)\,\xi_f A}{\left(E\left[\xi_f\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta}\right]\right)^2}, \tag{52}$$

where

$$A = E\left[\widetilde{\xi}\left(\widetilde{\xi}n + \xi_f n_f\right)^{-2-\delta}\right] E\left[\widetilde{\xi}\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta}\right] - E\left[\widetilde{\xi}^2\left(\widetilde{\xi}n + \xi_f n_f\right)^{-2-\delta}\right] E\left[\left(\widetilde{\xi}n + \xi_f n_f\right)^{-1-\delta}\right]. \tag{53}$$

After some algebra, $A$ can be rewritten as

$$\begin{aligned} A &= \frac{n_f}{n} E\left[\widetilde{\xi}\left(\widetilde{\xi}n+\xi_f n_f\right)^{-2-\delta}\right] E\left[\xi_f\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1-\delta}\right] \\ &\quad -\frac{n_f}{n} E\left[\xi_f\left(\widetilde{\xi}n+\xi_f n_f\right)^{-2-\delta}\right] E\left[\widetilde{\xi}\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1-\delta}\right]. \end{aligned} \tag{54}$$

Noticing that

$$E\left[\widetilde{\xi}\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1-\delta}\right] = \frac{p}{p_f} E\left[\xi_f\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1-\delta}\right], \tag{55}$$

$A$ can be rewritten as

$$A = \frac{n_f}{n} E\left[\xi_f\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1-\delta}\right] E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)\left(\widetilde{\xi}n+\xi_f n_f\right)^{-2-\delta}\right]. \tag{56}$$

Since

$$\frac{\partial\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1}}{\partial\widetilde{\xi}} < 0, \tag{57}$$

it follows from [12], Proposition 15 that

$$E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)\left(\widetilde{\xi}n+\xi_f n_f\right)^{-2-\delta}\right] < E\left[\left(\widetilde{\xi}-\frac{p}{p_f}\xi_f\right)\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1-\delta}\right] E\left[\left(\widetilde{\xi}n+\xi_f n_f\right)^{-1}\right] = 0. \tag{58}$$

Therefore, one can conclude that

$$n_f \gtreqless 0 \Leftrightarrow A \lesseqgtr 0 \Leftrightarrow \frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \lesseqgtr 0. \tag{59}$$

For case (ii), defining

$$n_f^{new} = n_f - \frac{a}{\xi_f}, \tag{60}$$

and following the same steps as above,

$$n_f^{new} \gtreqless 0 \Leftrightarrow n_f \lesseqgtr \frac{a}{\xi_f} \Leftrightarrow \frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \lesseqgtr 0. \tag{61}$$

For case (iii), defining

$$n_f^{new} = n_f + \frac{a}{\xi_f}, \tag{62}$$

and following the same steps as above,

$$n_f^{new} \gtreqless 0 \Leftrightarrow n_f \lesseqgtr -\frac{a}{\xi_f} \Leftrightarrow \frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \lesseqgtr 0. \tag{63}$$

For case (iv), we have $\tau'_A = 0$. Following an argument similar to that in Proposition 3, it can be easily verified that if $\tau'_A \geq 0$, the risky asset can never become urgently needed, or equivalently,

$$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} < 0. \tag{64}$$

■

**Remark 3** *For the Proposition 5(i) and (ii) utilities, it can be easily verified that $\tau'_A < 0$ and $\tau'_R \leq 0$. Therefore, we have $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} > 0$ when $n_f < 0$, which is consistent with Proposition 3(i). For the Proposition 5(i) CRRA (constant relative risk aversion) utility, it is well-known that the $MRS = \frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}$ is constant along each ray going through the origin in $n - n_f$ space. Therefore, it is not surprising that along the $n_f = 0$ ray, which is a horizontal ray starting from the origin, we have $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$. For the Proposition 5(ii) and (iii) utilities, the MRS is constant along rays through the translated origins.*

The geometric meaning of Proposition 5 can be illustrated by considering the Type (ii) preferences represented by eqn. (45). The $n - n_f$ plane in Figure 3 is divided by the $n_f = \frac{a}{\xi_f}$ horizontal line into two separate regions, which are characterized by different indifference curve properties. Above (below) this line, when moving horizontally to the right, the slope of the indifference curves becomes flatter (steeper).[18] Along the $n_f = \frac{a}{\xi_f}$ horizontal line in Figure 3, each of the indifference curves has the same slope $-\frac{E\left[\tilde{\xi}^{-\delta}\right]}{\xi_f E\left[\tilde{\xi}^{-1-\delta}\right]}$, implying that $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$. It should be noted that the same argument applies for Types (i) and (iii) except that the horizontal boundary lines correspond to $n_f = 0$ and $n_f = -\frac{a}{\xi_f}$, respectively.

**Remark 4** *It will be noted that for the HARA utility (45), the necessary and sufficient condition for the risky asset to be urgently needed is strikingly similar to the certainty Example 1. Indeed the corresponding induced Expected Utility function defined over assets parallels quite closely the non-standard certainty utility (5).*

## 3.4 General Homothetic Preferences

We next show that the MRS result (44) for the HARA Type (i) utility readily extends to general homothetic preferences whether or not they are representable by an Expected Utility function, where the term homothetic is defined as customary (see Deaton [10], pp.

[18]The utility defined by (45) has also been used to create Figure 2, where the movement from $P$ to $Q$ is in the region below the $n_f = \frac{a}{\xi_f}$ boundary line.

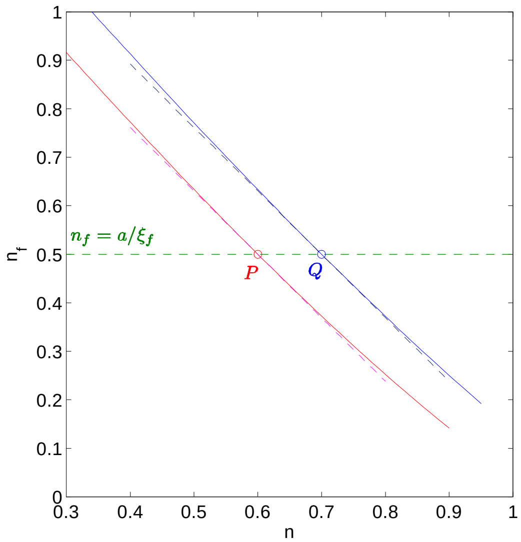



Figure 3:

143-5). It is not surprising the MRS is constant along the $n_f = 0$ ray since as discussed in Remark 3, for homothetic preferences the MRS is constant along each ray passing through the origin. Because in the following, preferences need not satisfy the standard NM axioms for the existence of an Expected Utility representation, we denote the utility defined over assets by $U(n, n_f)$ rather than $\mathcal{W}(n, n_f)$.

**Proposition 6** *Assuming preferences are homothetic and can be represented by* $U(n, n_f)$, *then*

$$\frac{\partial\left(\frac{U_n}{U_{n_f}}\right)}{\partial n} \gtreqless 0 \Leftrightarrow n_f \lesseqgtr 0. \tag{65}$$

**Proof.** Since preferences are homothetic, $\frac{U_n}{U_{n_f}}$ is a homogeneous function of degree zero. Therefore, along any ray $n_f = kn$, where $k$ is a constant, $\frac{U_n}{U_{n_f}}$ is a constant, implying that $\frac{\partial\left(\frac{U_n}{U_{n_f}}\right)}{\partial n} = 0$ if $n_f = 0$. Noticing that along each indifference curve $\frac{U_n}{U_{n_f}}$ decreases with $n$ and $n_f = kn$ is upward (downward) sloping for $n_f > (<)\, 0$, one can conclude that $\frac{\partial\left(\frac{U_n}{U_{n_f}}\right)}{\partial n} < (>)\, 0$ if $n_f > (<)\, 0$. Hence the result (65) holds. ■

## 3.5 Decreasing Relative Risk Aversion: Two Examples

For Proposition 5 Type (i)-(iii) HARA utilities and for general homothetic preferences, the value of $n_f$ clearly subdivides the $n-n_f$ asset space into two discrete regions corresponding to

$\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ being negative and positive. More generally it follows from Proposition 3(i) that if preferences exhibit decreasing relative risk aversion, the risky asset is always urgently needed in the portion of asset space where $n_f \leq 0$ and by continuity in at least some portion where $n_f > 0$. To characterize this latter region of asset space, we next consider two examples, where in each case the form of utility can be viewed as a natural extension of the Proposition 5(i) CRRA case. For the utility assumed in the first Example, it is instructive to compare the properties of $\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}$ to those of CRRA utility.

**Example 2** *Assume Expected Utility preferences characterized by the following NM index*

$$W(z) = -\left(\frac{z^{-\delta_1}}{\delta_1} + \frac{z^{-\delta_2}}{\delta_2}\right), \quad \delta_1, \delta_2 > -1. \tag{66}$$

*Computing $\tau_A$ and $\tau_R$ yields*

$$\tau_A(z) = -\frac{W''(z)}{W'(z)} = \frac{(1+\delta_1)z^{-\delta_1-2}}{z^{-\delta_1-1} + z^{-\delta_2-1}} + \frac{(1+\delta_2)z^{-\delta_2-2}}{z^{-\delta_1-1} + z^{-\delta_2-1}} \tag{67}$$

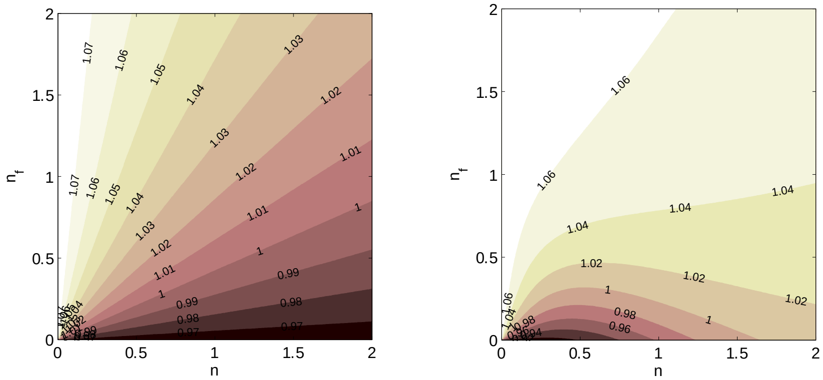
*and*

$$\tau_R(z) = -\frac{zW''(z)}{W'(z)} = \frac{(1+\delta_1)z^{-\delta_1-1}}{z^{-\delta_1-1} + z^{-\delta_2-1}} + \frac{(1+\delta_2)z^{-\delta_2-1}}{z^{-\delta_1-1} + z^{-\delta_2-1}}. \tag{68}$$

*It follows immediately that the utility (66) satisfies $\tau_A' < 0$ and $\tau_R' \leq 0$, where the equal sign can be reached if and only if $\delta_1 = \delta_2$. There are two senses in which the utility (66) can be viewed as an extension of CRRA utility. First, it takes the CRRA form as $\delta_1$ and $\delta_2$ converge. Second, the relative risk aversion for (66) is a weighted average of the relative risk aversion measures, $1+\delta_1$ and $1+\delta_2$, for two CRRA utilities corresponding to $\delta_1$ and $\delta_2$. And for this latter reason, (66) is referred to as weighted average constant relative risk aversion (WACRRA) utility. It follows from Proposition 3 that the risky asset is always urgently needed when $n_f < 0$. Therefore we focus on the region where $n_f \geq 0$ in the following analysis. For simplicity, consider a risky asset with payoff $\widetilde{\xi}$ that takes the values $\xi_{21}$ with probability $\pi_{21}$ and $\xi_{22}$ with the probability $\pi_{22} = 1 - \pi_{21}$. Without loss of generality, let $\xi_{21} > \xi_{22} > 0$. Suppose there exists a risk-free asset with payoff $\xi_f > 0$. Assume the following parameter values*

$$\xi_{21} = 1.2,\ \xi_{22} = 0.8,\ \xi_f = 1 \text{ and } \pi_{21} = 0.7. \tag{69}$$

*In Figure 4, we plot contours corresponding to constant values of the $MRS = \frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}$ for the positive orthant of asset space. The numbers on each contour correspond to different MRS values. For the $\delta_1 = \delta_2$ CRRA case in Figure 4(a), the constant MRS contours are rays starting from the origin which is consistent with preferences being homothetic. In Figure*




(a) $\delta_1 = \delta_2 = 2$ (b) $\delta_1 = 3.5, \delta_2 = 0.5$

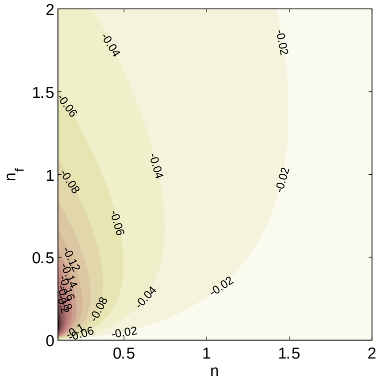
Figure 4:

*4(b) for the WACRRA utility where $\delta_1 > \delta_2$, the $n = 0$ positive vertical axis is a constant MRS contour, as in the CRRA Figure 4(a) case, where*

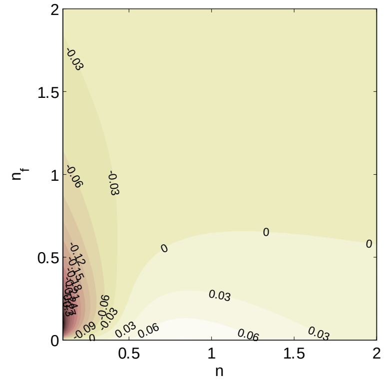
$$\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}} = \frac{E\widetilde{\xi}}{\xi_f} = 1.08. \tag{70}$$

*Also each MRS contour begins at the origin $(n, n_f) = (0, 0)$ as in the CRRA case. But corresponding to lower MRS values, the contours become more curved. Eventually as one moves along contours increasing $n$, the $n_f$ value decreases. If one considers a horizontal ray between $n_f = 0$ and $n_f = 0.4$, it is clear that as $n$ increases along the ray the MRS first declines and then increases implying that there is a point corresponding to $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$. Given the MRS contours in Figure 4, we next consider the pattern of changes in the MRS associated with increases in $n$. Contours corresponding to constant $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ values are displayed in Figure 5.*[19] *For the CRRA $\delta_1 = \delta_2$ utility in Figure 5(a), one always has $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \leq 0$, where the equal sign can be reached only along the $n_f = 0$ horizontal. This is consistent with Figure 4(a) and the conclusion of Proposition 5(i). The $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$ contour in Figure 5(b) corresponding to the WACRRA $\delta_1 > \delta_2$ utility forms the boundary between the region of positive and negative values of $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$. "Inside" the $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$ contour,*

[19] Note that the horizontal $n$-axis in Figure 5 does not start from 0 since when $n \to 0$, $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ becomes very negative. To illustrate the fine structure close to $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$, we let $n$ begin at 0.1. A similar argument applies to Figure 7(b) below.




(a) $\delta_1 = \delta_2 = 2$




(b) $\delta_1 = 3.5, \delta_2 = 0.5$

Figure 5:

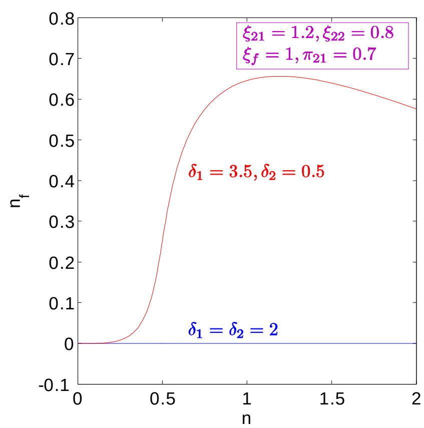
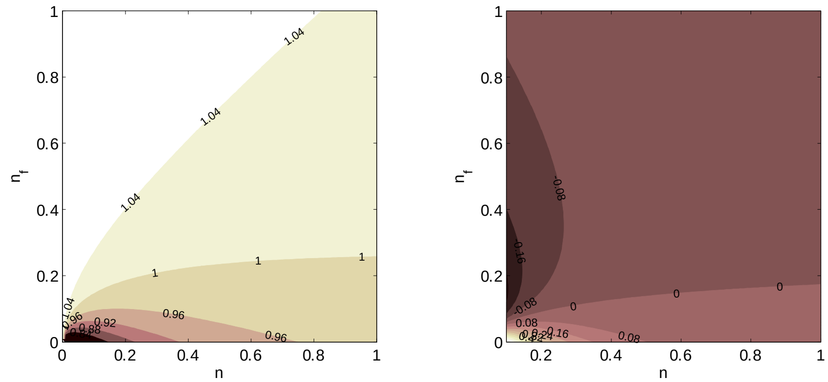



Figure 6:

*one always has* $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} > 0$ *with the risky asset being urgently needed and "outside" the* $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$ *contour, one always has* $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} < 0$*. This is consistent with the observation above that along any ray in Figure 4(b) between* $n_f = 0$ *and* $0.4$*, as* $n$ *increases the MRS value declines and then increases. To most clearly compare the CRRA and WACRRA* $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$ *boundaries of the region where the risky asset is urgently needed, see Figure 6.*

**Example 3** *Assume Expected Utility preferences characterized by the following Expo-Power*




(a) (b)

Figure 7:

*NM index*[20]

$$W(z) = -\exp\left(-\beta z^{\alpha}\right), \qquad \alpha, \beta \neq 0 \text{ and } \alpha\beta > 0. \tag{71}$$

*It can be easily verified that*

$$\tau_A(z) = -\frac{W''(z)}{W'(z)} = \frac{\alpha\left(\beta z^{\alpha} - 1\right) + 1}{z} \tag{72}$$

*and*

$$\tau_R(z) = -\frac{zW''(z)}{W'(z)} = \alpha\left(\beta z^{\alpha} - 1\right) + 1. \tag{73}$$

*The functional form (71) defines a family of utility functions corresponding to different values of the parameters $\alpha$ and $\beta$. It can easily be verified that absolute risk aversion is decreasing, constant or increasing if and only if $\alpha <, =, > 1$. Relative risk aversion is decreasing or increasing if and only if $\beta <$ or $> 0$.*[21] *To satisfy the conditions in Proposition 3(i), assume that $\alpha = -1$ and $\beta = -0.8$. This implies that the risky asset will always be urgently needed when $n_f \leq 0$ as well as for some region in the positive orthant of asset space. The parameter values (69) are also assumed to hold for this example. Paralleling the WACRRA case, contours corresponding to different $\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}$ and $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ values are plotted in Figure 7(a) and (b), respectively. It can be verified that the pattern and shape of the contours in Figures 7(a) and (b) are similar to those in Figure 4(b) and 5(b). The risky asset is urgently needed*

[20] The utility (71) was first introduced by Saha [31] and subsequently used in different applications by Abdellaoui, Barrios and Wakker [1] and Holt and Laury [17].

[21] Although it follows from the $\tau_R$ function (73) that relative risk aversion will be constant if $\beta = 0$, given that Saha [31] rules out the case where $\beta = 0$ in the definition of the family, it is necessary to modify the definition as done in Abdellaoui et al. [1].

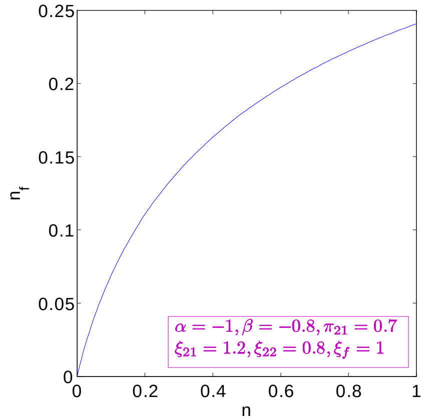



Figure 8:

*in the region inside the* $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$ *contour in Figure 7(b). To facilitate comparison with Figure 6, the* $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$ *contour has isolated in Figure 8.*[22]

# 4 Equilibrium Price and Risky Asset Supply

In this Section, we investigate the relationship between the equilibrium price ratio and the supply of the risky asset in a single ("representative") agent economy. We show that when (and only when) the risky asset is urgently needed, its equilibrium (relative) price increases with its supply. This seemingly counter intuitive equilibrium price behavior can be expected to occur more readily in uncertainty than in certainty settings because, as discussed in Section 3.2, under uncertainty a good is more likely to be urgently needed. Extending the results to economies with heterogeneous agents is straightforward for the case of aggregation (see the classic papers of Chipman [8] and Rubinstein [30]) and need not be discussed here.

Consider a standard single agent exchange economy setting, where the agent's preferences satisfy the assumptions in Subsection 3.1. Following [20] when solving the agent's demand problem, eqns. (10)-(11), one can think of fixing the budget constraint based on a given endowment and prices and finding the utility maximizing asset demands. On the other hand, when solving for equilibrium prices, one fixes the specific indifference curve passing through the endowment point and then solves for the equilibrium price ratio equal to the slope of the tangent to the indifference curve at that point. The optimal point corresponds

[22]It should be noted that unlike the WACRRA case in Figure 6, the $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$ contour in Figure 8 never declines with increasing $n$.

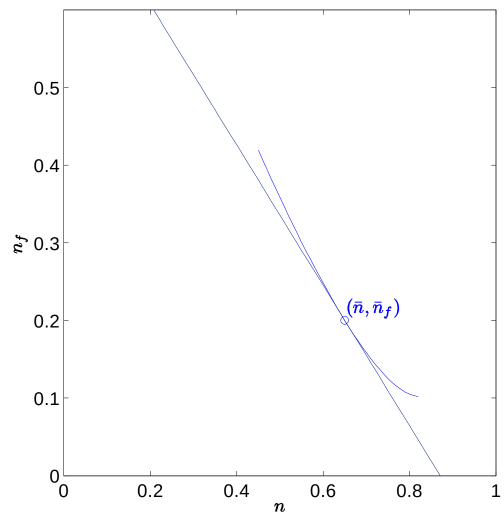



Figure 9:

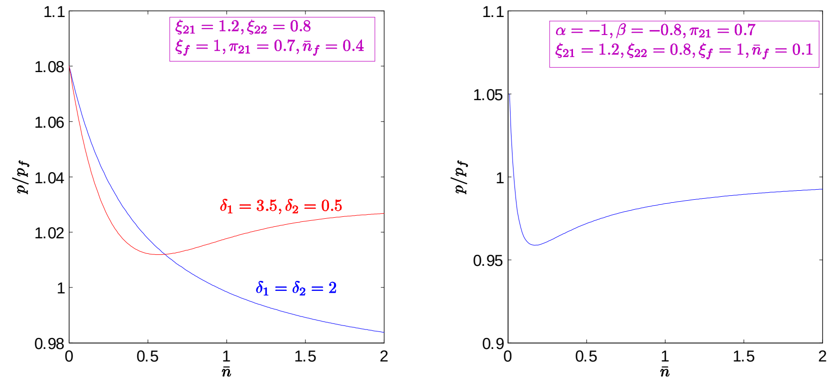
to the tangency point $(\overline{n}, \overline{n}_f)$ in Figure 9. Given the single agent setting, it is clear that there will be a unique equilibrium defined by $(p, p_f, n, n_f)$. This equilibrium corresponds to the fixed parameter set $(\overline{n}, \overline{n}_f, \widetilde{\xi}, \xi_f)$ where equilibrium prices are endogenous. Without loss of generality, we will use the risk free asset as the numeraire.

Since, as noted in Section 3.1, our assumption that $\frac{E\widetilde{\xi}}{p} > \frac{\xi_f}{p_f}$ implies $n > 0$, a positive endowment of the risky asset $\overline{n} > 0$ will be assumed, as is standard, throughout the remainder of this paper. On the other hand, we allow $\overline{n}_f \gtreqqless 0$, which runs contrary to the conventional assumption that the net supply of bonds is zero (e.g., [3]). In recent years, a number of papers have appeared which consider the cases of positive and negative net supplies of bonds (see footnote 3).[23] It should be noted that in much of this literature the assumption that $\overline{n}_f$ is positive, zero or negative is made for analytic convenience or to facilitate a particular dis-

[23] In [29], the authors summarize the argument for not requiring $\overline{n}_f = 0$ as follows

> The assumption that bonds are in zero net supply is consistent with an infinitely lived representative agent in an economy absent any frictions...By contrast, in a world with finitely lived investors, or with frictions, it may be possible for the current generation to borrow against the consumption of future generations, leading to a positive supply of bonds and risk-free consumption for the current generation over a significant time period. Indeed, in any economy in which Ricardian equivalence fails, government bonds can be in positive net supply. ([29], p. 3)

Cass and Pavlova [6] observe that while nonnegativity assumptions for commodity endowments are very defensible, there is nothing contradictory in dropping this assumption when considering financial assets, especially when there are no restrictions on asset trade.




(a) (b)

Figure 10:

cussion such as deficits. However, as we will see, the sign assumption on $\overline{n}_f$ can significantly impact whether the equilibrium price ratio $p/p_f$ increases or decreases with the supply of the risky asset.

Before giving our general result relating the risky asset's equilibrium (relative) price and its supply, we first consider the price-supply curve for the two Examples in Subsection 3.5. For the WACRRA and CRRA cases, assuming $\overline{n}_f = 0.4$ and the same parameter values (69), we plot the equilibrium price ratio $p/p_f$ versus $\overline{n}$ in Figure 10(a). It can be seen that for the $\delta_1 = \delta_2$ case, the equilibrium price ratio will always decrease with $\overline{n}$ and for the $\delta_1 > \delta_2$ case the price ratio will first decrease and then increase with $\overline{n}$. This is consistent with Figure 6 since if one draws a horizontal line at $n_f = 0.4$, it follows that (i) for $\delta_1 = \delta_2$ $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ is always negative and (ii) for $\delta_1 > \delta_2$ as $n$ increases the quantity $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ is at first negative and then becomes positive.[24] For the latter case, the zero slope point in Figure 10(a) corresponds to the point on the $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$ contour in Figure 6 where $(n, n_f) = (\overline{n}, \overline{n}_f)$. For the Expo-Power case, assuming $\overline{n}_f = 0.1, \alpha = -1, \beta = -0.8$ and the same parameter values (69), we plot the equilibrium price ratio $p/p_f$ versus $\overline{n}$ in Figure 10(b). As in the $\delta_1 > \delta_2$ WACRRA case, the price ratio first decreases and then increases with $\overline{n}$. This pattern is also consistent with the fact that in Figure 8 along the $n_f = 0.1$ horizontal, as

[24] It should be noted that in Figure 6 as $n$ increases, the $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} = 0$ contour eventually curves down and would intersect an $n_f = 0.4$ horizontal ray twice. Therefore, the equilibrium price ratio $p/p_f$ will decrease again when $\overline{n}$ is sufficiently large.

$n$ increases the value of $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n}$ is first negative and then becomes positive.[25] Given the above discussion, the seemingly puzzling price-supply behavior in Figures 10(a) and (b) can be easily explained by the risky asset being urgently needed as summarized by the following straightforward Proposition.[26]

**Proposition 7** *Assume a single agent exchange economy, where the optimization problem is given by eqns. (10)-(11). Then*

$$\frac{\partial\left(\frac{p}{p_f}\right)}{\partial \overline{n}} \gtreqless 0 \Leftrightarrow \frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} \gtreqless 0. \tag{74}$$

**Proof.** Given that $(n, n_f) = (\overline{n}, \overline{n}_f)$ in equilibrium, it follows from the first order condition that the resulting equilibrium price ratio is given by

$$\frac{p}{p_f} = \left.\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right|_{(n,n_f)=(\overline{n},\overline{n}_f)} = \frac{E\left[\widetilde{\xi} W'\left(\widetilde{\xi}\overline{n} + \xi_f \overline{n}_f\right)\right]}{E\left[\xi_f W'\left(\widetilde{\xi}\overline{n} + \xi_f \overline{n}_f\right)\right]}. \tag{75}$$

Hence eqn. (74) holds.[27] ■

**Remark 5** *Combining Propositions 1 and 7, it follows immediately that the price of a commodity or asset increases with its supply if and only if it's Engel curve is downward sloping. In the case of commodities where quantities are always assumed to be positive, this is equivalent to a good being inferior. This is consistent with the argument of Nachbar [27] that in a multigood setting, price can increase with supply only if the composite commodity formed by the other commodities is inferior. In the case of assets, as discussed above, an asset is normal if the quantity is positive and increases with income or negative and decreases with income. Thus it follows from Proposition 7 that whether the risky asset's (relative) price increases with its supply does not depend on whether the risk free asset is an inferior good*

[25] It should be noted that when $\tau'_A < 0$ and $\tau'_R > 0$, it is not possible to conclude from Proposition 3(ii) whether or not $\frac{\partial\left(\frac{\mathcal{W}_n}{\mathcal{W}_{n_f}}\right)}{\partial n} > 0$ in the $n_f < 0$ portion of asset space. However Proposition 5(iii) provides one example where this is the case. It is possible to create an another example utilizing the Expo-Power utility. Assuming $\alpha = \beta = 0.5$, it is straightforward to show that there exists a region of the $n_f < 0$ portion of asset space in which the risky asset is urgently needed and for a single agent economy the equilibrium price ratio $p/p_f$ increases with $\overline{n}$.

[26] Also see [22].

[27] If an equilibrium exists, the no arbitrage condition is automatically satisfied and the no bankruptcy condition in the equilibrium setting is given by $\inf \widetilde{\xi}\overline{n} + \xi_f \overline{n}_f > 0$. Note that for some special forms of utility such as (45), the no bankruptcy condition needs to be modified. (For example in this case, the condition is given by $\inf \widetilde{\xi}\overline{n} + \xi_f \overline{n}_f > a$.)

*but rather whether it's Engel curve is decreasing with income or the risky asset is urgently needed.*[28]

Combining Propositions 2-6 with 7, one can obtain alternative conditions for when the equilibrium price ratio $p/p_f$ increases with the risky asset supply $\overline{n}$. In the cases of Propositions 5 and 6, the conditions depend solely on the supply of the risk free asset $\overline{n}_f$. For example, combining Propositions 5 and 7, one obtains the following very simple necessary and sufficient condition.

**Corollary 1** *Assume a single agent exchange economy, where the optimization problem is given by eqns. (10)-(11). Let the agent's NM index $W(z)$ be given by*

$$W(z) = -\frac{(z-a)^{-\delta}}{\delta}, \tag{76}$$

*where $\delta > -1$ and $a$ is allowed to be negative, zero or positive. Then*

$$\frac{\partial\left(\frac{p}{p_f}\right)}{\partial\overline{n}} \gtreqqless 0 \Leftrightarrow \overline{n}_f \lesseqqgtr \frac{a}{\xi_f}. \tag{77}$$

This result covers the constant, decreasing and increasing relative risk aversion members of the HARA class corresponding respectively to the Type (i), (ii) and (iii) utilities in Proposition 5. For Type (ii) where $a > 0$, if one assumes, as is standard, a zero supply of the risk free asset, then it is always the case that equilibrium price ratio $p/p_f$ increases with the supply of the risky asset.

# 5 Concluding Comments

In certainty commodity choice problems where utility is assumed to be supermodular and concave, it is not possible for a good to be urgently needed. However in the classic two asset uncertainty setting, the risky asset can become urgently needed and its equilibrium (relative) price can increase with supply even when preferences are represented by popular members of the HARA class of Expected Utility functions. The Arrow-Pratt risk aversion measures and the supply of the risk free asset play important roles in explaining this behavior.

Throughout most of our analysis, we assume Expected Utility preferences. One exception is in Proposition 6, where we prove that the risky asset can be urgently needed if preferences are homothetic whether or not they are representable by an Expected Utility function. This raises the very interesting question of whether it is possible to find alternative conditions

[28] Also see the discussion of Kohli [21] in a two commodity, certainty distribution economy setting where one commodity is assumed to be a Giffen good.

to those in Propositions 3 and 4 for non-Expected Utility preferences which result in the risky asset being urgently needed and the (relative) price of the risky asset increasing with its supply.

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